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On the Positivity of the Fundamental Polynomials for Generalized Hermite–Fejér Interpolation on the Chebyshev Nodes

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It is shown that the fundamental polynomials for $(0, 1, ..., 2m+1)$ Hermite–Fejér interpolation on the zeros of the Chebyshev polynomials of the first kind are non-negative for $-1 \leqslant x \leqslant 1$, thereby generalising a well-known property of the original Hermite–Fejér interpolation method. As an application of the result, Korovkin's theorem on monotone operators is used to present a new proof that the $(0, 1, ..., 2m+1)$ Hermite–Fejér interpolation polynomials of $f \in C[-1, 1]$, based on n Chebyshev nodes, converge uniformly to f as $n \to \infty$. © 1999 Academic Press

1. INTRODUCTION

Suppose f is a continuous real-valued function defined on the interval $[-1, 1]$, and let

$$X = \{x_{k,n} \colon k = 1, 2, ..., n;\ n = 1, 2, 3, ...\}$$

be a triangular matrix such that for all n,

$$1 \geqslant x_{1,n} > x_{2,n} > \cdots > x_{n,n} \geqslant -1.$$

Then, for each integer $m \geqslant 0$, there exists a unique polynomial $H_{m,n}(X, f, x)$ of degree at most $(m+1)n-1$ which satisfies

$$H^{(r)}_{m,n}(X, f, x_{k,n}) = \delta_{0,r}\, f(x_{k,n}), \qquad 1 \leqslant k \leqslant n, \quad 0 \leqslant r \leqslant m.$$

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$H_{m,n}(X, f, x)$ is referred to as the $(0, 1, ..., m)$ Hermite–Fejér (HF) interpolation polynomial of $f(x)$, and it can be expressed as

$$H_{m,n}(X, f, x) = \sum_{k=1}^{n} f(x_{k,n}) A_{k,m,n}(X, x),$$

where $A_{k,m,n}(X, x)$ is the unique polynomial of degree at most $(m+1)n-1$ such that

$$A^{(r)}_{k,m,n}(X, x_{j,n}) = \delta_{0,r}\delta_{k,j}, \qquad 1 \leqslant k, j \leqslant n, \quad 0 \leqslant r \leqslant m.$$

The $A_{k,m,n}(X, x)$ are referred to as the fundamental polynomials for $(0, 1, ..., m)$ HF interpolation on X, and the function

$$\lambda_{m,n}(X, x) = \sum_{k=1}^{n} |A_{k,m,n}(X, x)|,$$

which is the Lebesgue function for $(0, 1, ..., m)$ HF interpolation on X, plays a fundamental role in discussion of the convergence of $H_{m,n}(X, f, x)$ to $f(x)$ as $n \to \infty$.

Now, $H_{0,n}(X, f, x)$ is the well-known Lagrange interpolation polynomial of $f(x)$. A classic result of Faber [3] states that for any matrix X, there exists $f \in C[-1, 1]$ so that $H_{0,n}(X, f, x)$ does not tend uniformly to $f(x)$ on $[-1, 1]$ as $n \to \infty$. The initial motivation for considering $(0, 1, ..., m)$ HF interpolation for $m \geqslant 1$ came from Fejér's result [4] that if T denotes the matrix of Chebyshev nodes,

$$T = \left\{ \cos\left(\frac{2k-1}{2n}\pi\right): k = 1, 2, ..., n; n = 1, 2, 3, ... \right\},$$

and if $f \in C[-1, 1]$, then $H_{1,n}(T, f, x) \to f(x)$ uniformly in $[-1, 1]$. Thus the $(0, 1)$ HF process has better convergence properties than the Lagrange method, at least on the node system T.

For general $(0, 1, ..., m)$ HF interpolation, Szabados [10] showed that if m is even, then for any X there exists $f \in C[-1, 1]$ so that $H_{m,n}(X, f, x)$ does not tend uniformly to $f(x)$ on $[-1, 1]$ as $n \to \infty$. However, if m is odd, Sakai [9] and Vértesi [11] obtained the following result.

THEOREM A. *If m is odd and $f \in C[-1, 1]$, then $H_{m,n}(T, f, x) \to f(x)$ uniformly in $[-1, 1]$ as $n \to \infty$.*

(Vértesi's result is actually more general than this, as he obtained uniform convergence results for interpolation on the zeros of various Jacobi polynomials, including the Chebyshev polynomials. This work has been subsequently developed and sharpened in terms of so-called ρ-normal matrices X; see Vértesi [12].) These results illustrate the principle that even-order HF processes tend to have similar properties to the Lagrange interpolation method, while odd-order HF processes are akin to the original Hermite–Fejér method (i.e., to (0, 1) HF interpolation).

An important property of (0, 1) HF interpolation on the Chebyshev nodes is that the fundamental polynomials $A_{k,1,n}(T, x)$ are non-negative for $-1 \leqslant x \leqslant 1$. A discussion of this result and its subsequent use in the original proof of Fejér's uniform convergence result for the polynomials $H_{1,n}(T, f, x)$ is given by Rivlin [8, Sect. 1.4]. However, there is a simpler means of establishing Fejér's result, which uses Korovkin's theorem on monotone operators to take particular advantage of the non-negativity of the fundamental polynomials. This approach is due to Korovkin himself, and is discussed in, for example, Cheney [1, Sect. 3.3].

For (0, 1, 2, 3) HF interpolation, it is also known that the fundamental polynomials $A_{k,3,n}(T, x)$ are non-negative for $-1 \leqslant x \leqslant 1$. This result is due to Kryloff and Stayermann [6]. (See also Laden [7].) As a consequence of the non-negativity of the $A_{k,m,n}(T, x)$ for $m = 1, 3$, the Lebesgue functions $\lambda_{m,n}(T, x)$ are given by $\lambda_{m,n}(T, x) = \sum_{k=1}^{n} A_{k,m,n}(T, x)$. The right-hand side is a polynomial of degree at most $(m+1)\,n-1$ which has value 1 and vanishing first m derivatives at each of the n nodes, and since such a polynomial is uniquely determined, it follows that if $m = 1, 3$, then $\lambda_{m,n}(T, x) = 1$ for all x.

In this paper we generalize these results for (0, 1) and (0, 1, 2, 3) HF interpolation to higher odd-order HF interpolation, as follows.

THEOREM 1. *If m is odd, the fundamental polynomials $A_{k,m,n}(T, x)$ for $(0, 1, ..., m)$ HF interpolation on the Chebyshev nodes T satisfy*

$$A_{k,m,n}(T, x) \geqslant 0, \qquad -1 \leqslant x \leqslant 1. \tag{1}$$

COROLLARY. *If m is odd, the Lebesgue function $\lambda_{m,n}(T, x)$ satisfies*

$$\lambda_{m,n}(T, x) \equiv 1. \tag{2}$$

Note that (2) improves Sakai's result [9, Lemma 3] that for odd m, $\lambda_{m,n}(T, x)$ is uniformly bounded with respect to $x \in [-1, 1]$ and n.

The proof of Theorem 1 is presented in Section 2. In Section 3, the non-negativity of the $A_{k,m,n}(T, x)$ is used in conjunction with Korovkin's theorem on monotone operators to present a new proof of Theorem A.

2. PROOF OF THEOREM 1

Suppose m is odd. By a result of Kreß [5, Theorem 1.1] there exists a unique trigonometric polynomial $T_{m,n}(\theta)$ of the form

$$T_{m,n}(\theta)=\sum_{k=0}^{(m+1)n-1} a_k\cos k\theta+\sum_{k=1}^{(m+1)n} b_k\sin k\theta$$

so that

$$T_{m,n}^{(r)}\left(\frac{j\pi}{n}\right)=\delta_{0,r}\delta_{0,j},\qquad 0\leqslant r\leqslant m,\ 0\leqslant j\leqslant 2n-1.$$

Because $(T_{m,n}(\theta)+T_{m,n}(-\theta))/2$ satisfies the same conditions as does $T_{m,n}(\theta)$, it follows by the uniqueness property of $T_{m,n}$ that it is even, and so

$$T_{m,n}(\theta)=\sum_{k=0}^{(m+1)n-1} a_k\cos k\theta.$$

We now show that $T_{m,n}$ is non-negative. In the interval $[0,2\pi)$, $T'_{m,n}(\theta)$ has zeros of order m at $j\pi/n$, $0\leqslant j\leqslant 2n-1$, and (by Rolle's Theorem) has a zero in each interval $(j\pi/n,(j+1)\pi/n)$ for $1\leqslant j\leqslant 2n-2$. This identifies $2(m+1)n-2$ zeros of $T'_{m,n}(\theta)$ in $[0,2\pi)$, and since $T'_{m,n}(\theta)$ has degree $(m+1)n-1$, we have identified all zeros of $T'_{m,n}(\theta)$ in $[0,2\pi)$. Note that all the zeros of $T'_{m,n}(\theta)$ are of odd order, and so correspond to turning points of $T_{m,n}(\theta)$. The conditions $T_{m,n}(0)=1$ and $T_{m,n}(j\pi/n)=0$, $1\leqslant j\leqslant 2n-1$, then imply that $T_{m,n}(\theta)\geqslant 0$ for all θ.

Next put $\theta_{j,n}=(2j-1)\pi/(2n)$, and for $1\leqslant k\leqslant n$, consider

$$t_{k,m,n}(\theta)=T_{m,n}(\theta-\theta_{k,n})+T_{m,n}(\theta+\theta_{k,n}).$$

Since $T_{m,n}(\theta)$ is even, it follows that $t_{k,m,n}(\theta)$ is even, and so it is a non-negative cosine polynomial of degree $(m+1)n-1$ which satisfies

$$t_{k,m,n}^{(r)}(\theta_{j,n})=\delta_{0,r}\delta_{k,j},\qquad 0\leqslant r\leqslant m,\quad 1\leqslant j\leqslant n.$$

The result (1) then follows from the observation that

$$A_{k,m,n}(T,x)=t_{k,m,n}(\cos^{-1}x).$$

3. NEW PROOF OF THEOREM A

Suppose m is odd. Because of the non-negativity of the $A_{k,m,n}(T, x)$ on $[-1, 1]$, Theorem A will follow from Korovkin's theorem on monotone operators if it can be shown that

(i) $H_{m,n}(T, 1, x) \to 1$ uniformly in x on $[-1, 1]$ as $n \to \infty$,

(ii) $H_{m,n}(T, \phi_x, x) \to 0$ uniformly in x on $[-1, 1]$, where $\phi_x(t) = (x-t)^2$.

With regard to (i), observe that $H_{m,n}(T, 1, x)$ is the unique polynomial of degree at most $(m+1)\,n-1$ which has value 1 and vanishing first m derivatives at each of the n nodes, so $H_{m,n}(T, 1, x) = 1$ for all x. Thus (i) is immediately satisfied.

For (ii), we need to show that if $\theta_{k,n} = (2k-1)\,\pi/(2n)$, $x_{k,n} = \cos\theta_{k,n}$, and

$$\Psi_{m,n}(x) = \sum_{k=1}^{n} (x - x_{k,n})^2 A_{k,m,n}(T, x),$$

then $\Psi_{m,n}(x) \to 0$ uniformly on $[-1, 1]$ as $n \to \infty$. Explicit formulas for the $A_{k,m,n}(T, x)$ are, in general, quite complicated. However, Vértesi [11, Lemma 3.11] has shown that for fixed m, there exists a constant c so that if we write

$$A_{k,m,n}(T, x) = (A_{k,0,n}(T, x))^{m+1} \sum_{i=0}^{m} e_{i,k,m,n}(x - x_{k,n})^i,$$

then

$$|e_{i,k,m,n}| \leqslant \begin{cases} c\left(\dfrac{n}{\sin\theta_{k,n}}\right)^i & \text{if} \quad i = 0, 2, 4, ..., m-1, \\[2ex] c\,\dfrac{n^{i-1}}{\sin^{i+1}\theta_{k,n}} & \text{if} \quad i = 1, 3, 5, ..., m. \end{cases}$$

Thus

$$|\Psi_{m,n}(x)| \leqslant c \sum_{k=1}^{n} (A_{k,0,n}(T, x))^{m+1} (x - x_{k,n})^2 \left(1 + \frac{|x - x_{k,n}|}{\sin^2\theta_{k,n}}\right) \times \sum_{i=0}^{(m-1)/2} \left(\frac{n(x - x_{k,n})}{\sin\theta_{k,n}}\right)^{2i}.$$

If $T_n(x)$ denotes the nth Chebyshev polynomial (i.e., $T_n(x) = \cos(n \cos^{-1} x)$, $x \in [-1, 1]$), then

$$A_{k,0,n}(T, x) = \frac{(-1)^{k-1} T_n(x) \sin \theta_{k,n}}{n(x - x_{k,n})}.$$

(See, for example, Rivlin [8, Sect. 1.3].) Thus

$$|\Psi_{m,n}(x)| \leqslant c \frac{(T_n(x))^2}{n^2} \sum_{k=1}^{n} (A_{k,0,n}(T, x))^{m-1} (\sin^2 \theta_{k,n} + |x - x_{k,n}|) \times \sum_{i=0}^{(m-1)/2} \left(\frac{n(x - x_{k,n})}{\sin \theta_{k,n}}\right)^{2i}.$$

Now, if $-1 \leqslant x \leqslant 1$, then $\sin^2\theta_{k,n} + |x - x_{k,n}| < 3$. Also, by a result of Erdős and Grünwald [2], if $x \in [-1, 1]$ then $|A_{k,0,n}(T, x)| < 4/\pi$ for all k and n. Thus, for $0 \leqslant i \leqslant (m-1)/2$, we have

$$(A_{k,0,n}(T, x))^{m-1} \left(\frac{n(x - x_{k,n})}{\sin \theta_{k,n}}\right)^{2i} = (A_{k,0,n}(T, x))^{m-1-2i} (T_n(x))^{2i} < \left(\frac{4}{\pi}\right)^{m-1},$$

and so

$$|\Psi_{m,n}(x)| \leqslant \frac{3c(m+1)}{2n} \left(\frac{4}{\pi}\right)^{m-1}.$$

Thus $\Psi_{m,n}(x) \to 0$ uniformly on $[-1, 1]$ as $n \to \infty$, and so (ii) is established.

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